The ABC of Hyper Recursions

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Dedicated to Roderick Wong on occasion of his $60^{\rm th}$ birthday.

Abstract

Each family of Gauss hypergeometric functions

$$f_n = {}_2F_1(a+\varepsilon_1 n, b+\varepsilon_2 n; c+\varepsilon_3 n; z),$$

for fixed $\varepsilon_j = 0, \pm 1$ (not all ε_j equal to zero) satisfies a second order linear difference equation of the form

$$A_n f_{n-1} + B_n f_n + C_n f_{n+1} = 0.$$

Because of symmetry relations and functional relations for the Gauss functions, many of the 26 cases (for different ε_j values) can be transformed into each other. We give a set of basic equations from which all other equations can be obtained. For each basic equation, we study the existence of minimal solutions and the character of f_n (minimal or dominant) as $n \to \pm\infty$. A second independent solution is given in each basic case which is dominant when f_n is minimal and vice-versa. In this way, satisfactory pairs of linearly independent solutions for each of the 26 second order linear difference equations can be obtained.

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1 Introduction

The Gauss hypergeometric functions

$$f_n = {}_2F_1\left(\begin{matrix} a+\varepsilon_1 n,\ b+\varepsilon_2 n \\ c+\varepsilon_3 n \end{matrix}; z\right), \tag{1.1}$$

where

$${}_2F_1\left(\begin{matrix} a,\ b \\ c \end{matrix}; z\right) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n\, n!} z^n, \quad |z| < 1 \tag{1.2}$$

satisfy three-term recurrence relations of the form

$$A_n f_{n-1} + B_n f_n + C_n f_{n+1} = 0. \tag{1.3}$$

Examples are given in [1, p. 558].

In this paper we consider recursions with respect to n for the cases

$$\varepsilon_j = 0, \pm 1, \quad j = 1,2,3, \quad \varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 \neq 0. \tag{1.4}$$

and we study the condition of these recurrences by obtaining the regions in the z-complex plane where a minimal solution exists.

A solution f_n of the recurrence relation (1.3) is said to be *minimal* if there exists a linearly independent solution g_n, of the same recurrence relation such that $f_n/g_n \to 0$ as $n \to \infty$. In that case g_n is called a *dominant* solution. When a recurrence admits a minimal solution (unique up to a constant factor), this solution should be included in any numerically satisfactory pair of solutions of the recurrence. Given a solution of the recurrence, it is crucial to know the character of the solution (minimal, dominant or none of them) in order to apply the recurrence relation in a numerically stable way. Indeed, if f_n is minimal as $n \to +\infty$, forward recurrence (increasing n) is an ill conditioned process because small initial errors will generally dominate the recursive solution by introducing an initially small component of a dominant solution; backward recurrence is well conditioned in this case. The opposite situation takes place for dominant solutions.

The problem of determining the z-values for which the f_n functions are minimal or dominant is considered in this paper. For the different recursion formulas we give a companion solution to f_n which, together with f_n, form a numerically satisfactory pair of the corresponding three-term recurrence relation.

2 Basic recursion relations and their solutions

There are 26 recursion relations for these functions for all choices of ε_j. However, we can use several functional relations in order to reduce our study to few basic recursions equations. First, we have the symmetry relation

$${}_2F_1\left(\begin{matrix} a,\ b \\ c \end{matrix}; z\right) = {}_2F_1\left(\begin{matrix} b,\ a \\ c \end{matrix}; z\right). \tag{2.1}$$

In addition, the following relations can be used

$$ {}_2F_1\left(\begin{matrix} a,\ b \\ c \end{matrix}; z\right) = (1-z)^{-a}\, {}_2F_1\left(\begin{matrix} a,\ c-b \\ c \end{matrix}; \frac{z}{z-1}\right), \tag{2.2} $$

$$ {}_2F_1\left(\begin{matrix} a,\ b \\ c \end{matrix}; z\right) = (1-z)^{-b}\, {}_2F_1\left(\begin{matrix} c-a,\ b \\ c \end{matrix}; \frac{z}{z-1}\right), \tag{2.3} $$

$$ {}_2F_1\left(\begin{matrix} a,\ b \\ c \end{matrix}; z\right) = (1-z)^{c-a-b}\, {}_2F_1\left(\begin{matrix} c-a,\ c-b \\ c \end{matrix}; z\right). \tag{2.4} $$

See [1, p. 559]. These relations show that a great number of the 26 cases are equivalent or follow from each other. In fact we can find 5 basic forms that need to be studied, while the remaining cases follow from the relations in (2.1)–(2.4). Few cases for which $\varepsilon_j = 2$ will be treated: only those which follow directly from the 26 cases for which $|\varepsilon_j| \le 1$, $j = 1, 2, 3$.

When $c = 0, -1, -2, \ldots$ the Gauss hypergeometric function in (1.2) is not defined. Also, when a or b assume non-positive integer values, the series in (1.2) terminates. In the following we will not distinguish about these special cases and the general cases, and we assume that all representations of the functions to be given are well defined, and we will not specify that special values of the parameters should be excluded in the results.

In Table 1 we give an overview of all possible recursions, and indicate the five basic forms. Observe that we take recursion in positive direction equivalent with recursion in negative direction, however we will need to distinguish between both directions when studying the asymptotic behavior of the solutions; see Section 4. Apart from the notation in Table 1, we will also use the notation $(\mathrm{sign}(\varepsilon_1)\,\mathrm{sign}(\varepsilon_2)\,\mathrm{sign}(\varepsilon_2))$ when $|\varepsilon_j| \le 1$; for instance, the case $k = 2$ will be also represented as $(+\,+\,0)$.

2.1 Selection of a second solution

Once we have reduced the number of basic recursions to be studied to 5, we will give for each basic form the coefficients A_n, B_n and C_n of the recursion relation (1.3) and we will study the character of f_n as a solution of the corresponding recurrence relation. For this, we will need to find a second solution of the relation which forms a satisfactory pair of solutions together with f_n (i.e. a pair which includes the minimal solution when it exists).

This second solution is chosen from several connection formulas between the hypergeometric functions. For example, the functions

$$ {}_2F_1\left(\begin{matrix} a,\ b \\ c \end{matrix}; z\right) \quad \text{and} \quad z^{1-c}\, {}_2F_1\left(\begin{matrix} a-c+1,\ b-c+1 \\ 2-c \end{matrix}; z\right) \tag{2.5} $$

satisfy the same differential equation; see [9, p. 112]. Both functions in (2.5) also satisfy the same difference equation, when a suitable normalization for the

Table 1: Only 5 basic forms remain

k	ε_1	ε_2	ε_3	type	comments
1	1	1	1	$\equiv 13$	use (2.4)
2	1	1	0	basic form	
3	1	1	-1	basic form	
4	1	0	1	$\equiv 13$	use (2.3)
5	1	0	0	basic form	
6	1	0	-1	basic form	
7	1	-1	1	$\equiv 16$	use (2.3)
8	1	-1	0	$\equiv 2$	use (2.2)
9	1	-1	-1	$\equiv 6$	use (2.2)
10	0	1	1	$\equiv 13$	use (2.2)
11	0	1	0	$\equiv 5$	use (2.1)
12	0	1	-1	$\equiv 6$	use (2.1)
13	0	0	1	basic form	
14	0	0	0		void
15	0	0	-1	$\equiv 13$	change sign in 13
16	0	-1	1	$\equiv 12$	change signs in 12
17	0	-1	0	$\equiv 11$	use (2.2)
18	0	-1	-1	$\equiv 15$	use (2.2)
19	-1	1	1	$\equiv 7$	use (2.1)
20	-1	1	0	$\equiv 8$	use (2.1)
21	-1	1	-1	$\equiv 9$	use (2.1)
22	-1	0	1	$\equiv 16$	use (2.1)
23	-1	0	0	$\equiv 17$	use (2.1)
24	-1	0	-1	$\equiv 18$	use (2.1)
25	-1	-1	1	$\equiv 3$	change signs in 3
26	-1	-1	0	$\equiv 2$	use (2.4)
27	-1	-1	-1	$\equiv 15$	use (2.4)

second function is chosen. To obtain this normalization we use the relation

$$
\begin{aligned}
{}_2F_1\left({a,\ b \atop c}\,;\,z\right) \quad &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\,{}_2F_1\left({a,\ b \atop a+b-c+1}\,;\,1-z\right) \\
&+ \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}\,(1-z)^{c-a-b}\,{}_2F_1\left({c-a,\ c-b \atop c-a-b+1}\,;\,1-z\right)
\end{aligned}
\tag{2.6}
$$

(see see [1, Eq. 15.3.6] or [9, p. 113]). By replacing z by $1-z$ and $a+b-c+1$ by c we can write this in the form

$$
\begin{aligned}
{}_2F_1\left({a,\ b \atop c}\,;\,z\right) &= P\,{}_2F_1\left({a,\ b \atop a+b-c+1}\,;\,1-z\right) \\
&- Q\,z^{1-c}\,{}_2F_1\left({a-c+1,\ b-c+1 \atop 2-c}\,;\,z\right),
\end{aligned}
\tag{2.7}
$$

where

$$
P = \frac{\Gamma(a+1-c)\,\Gamma(b+1-c)}{\Gamma(a+b-c+1)\,\Gamma(1-c)}, \quad Q = \frac{\Gamma(c-1)\,\Gamma(a+1-c)\,\Gamma(b+1-c)}{\Gamma(a)\,\Gamma(b)\,\Gamma(1-c)}.
\tag{2.8}
$$

Another connection formula to be used is

$$
\begin{aligned}
&{}_2F_1\left({a,\ b \atop c}\,;\,z\right) = \\
&\frac{\Gamma(1-a)\Gamma(b-c+1)}{\Gamma(1-c)\Gamma(b-a+1)}(-z)^{a-c}(1-z)^{c-a-b}{}_2F_1\left({1-a,\ c-a \atop b-a+1}\,;\,\frac{1}{z}\right) \\
&-\frac{\Gamma(c-1)\Gamma(b-c+1)\Gamma(1-a)}{\Gamma(b)\Gamma(c-a)\Gamma(1-c)}(-z)^{1-c}(1-z)^{c-a-b}{}_2F_1\left({1-a,\ 1-b \atop 2-c}\,;\,z\right),
\end{aligned}
\tag{2.9}
$$

where $|\mathrm{phase}(-z)| < \pi$. This relation follows from [1, Eq. 15.3.7], after we change $z \to 1/z$ and use the result (2.4).

We can take one of the two terms in the right-hand sides of (2.6), (2.7) and (2.9) as a second solution of the equation that is satisfied by the function in the left-hand side. We need to verify if this second solution is linearly independent of the first solution.

When selecting one of these candidates for the second solution, we can skip the gamma functions and other terms that are constant in the recursion. Also, gamma functions of the form $\Gamma(a-n)$ will be replaced by using the relation (see [9, p. 74])

$$
\Gamma(a-n) = \frac{(-1)^n\pi}{\sin \pi a \Gamma(n+1-a)}.
\tag{2.10}
$$

2.2 Basic forms and their solutions

Now we summarize the main results that will be obtained for each basic recursion. We provide the coefficients of the recursion as well as a satisfactory pair of independent solutions, giving information on the character of the f_n functions. In later sections we will obtain the regions of existence of minimal solutions by means of Perron theorem and we will identify the minimal and a dominant solution by analyzing the asymptotic behaviour of well chosen solutions.

2.2.1 Basic form $k = 2$

The $(++0)$ recursion relation reads

$$A_2(a+n,b+n)y_{n-1} + B_2(a+n,b+n)y_n + C_2(a+n,b+n)y_{n+1} = 0, \quad (2.11)$$

where

$$\begin{aligned} A_2(a,b) &= (c-a)(c-b)(c-a-b-1), \\ B_2(a,b) &= (c-a-b)\{c(a+b-c)+c-2ab+ \\ &\quad +z[(a+b)(c-a-b)+2ab+1-c]\}, \\ C_2(a,b) &= ab(c-a-b+1)(1-z)2, \end{aligned} \quad (2.12)$$

with solutions given by

$$\begin{aligned} f_n &= {}_2F_1\left({a+n,\ b+n \atop c};\, z\right), \\ g_n &= \frac{\Gamma(a+n+1-c)\,\Gamma(b+n+1-c)}{\Gamma(a+b+2n-c+1)}\,{}_2F_1\left({a+n,\ b+n \atop a+b+2n-c+1};\, 1-z\right). \end{aligned} \quad (2.13)$$

The second solution is taken from the first term in Eq. (2.7).

As we will later show, when $z \le 0$ the recurrence has no minimal solutions, whereas in compact domains that do not contain points of $(-\infty, 0]$, f_n is a dominant solution and g_n is minimal.

This case has applications for Jacobi polynomials. We have

$$P_n^{(\alpha,\beta)}(x) = \binom{n+\alpha}{n}\left(\frac{1+x}{2}\right)^n {}_2F_1\left({-n,\ -\beta-n \atop \alpha+1};\, z\right), \quad z = \frac{x-1}{x+1}. \quad (2.14)$$

A representation with $+n$ at the a and b places follows from applying (2.4). In the interval of orthogonality $-1 \le x \le 1$, we have $z \le 0$, and if $x \in [-1,1]$ the recursion relation of the Jacobi polynomials can be used for computing these functions in forward direction. Only the usual rounding errors should be taken into account.

Notice also how, in addition, information iis obtained regarding the recursion satisfied by

$$f_n = {}_2F_1\left({a+n,\ b+n \atop c+2n};\, z\right). \quad (2.15)$$

By replacing z by $1-z$ and c by $a+b-c$ (see also Eq. (2.10)) we see that

$$g_n = \frac{\Gamma(c+2n)}{\Gamma(n+1-b+c)\Gamma(n+1-a+c)}\,{}_2F_1\left({a+n,\ b+n \atop a+b-c};1-z\right) \tag{2.16}$$

is also a solution of the same recurrence, f_n being minimal when z is in compact domains not containing points in the interval $[1,+\infty)$.

2.2.2 Basic form $k=3$

The $(++-)$ recursion relation reads

$$\begin{aligned} &A_3(a+n,b+n,c-n)y_{n-1}+B_3(a+n,b+n,c-n)y_n+\\ &\qquad C_3(a+n,b+n,c-n)y_{n+1}=0, \end{aligned} \tag{2.17}$$

where

$$\begin{aligned} A_3(a,b,c) &= -(a-c)(a-c-1)(b-1-c)(b-c)zU,\\ B_3(a,b,c) &= c[c_1U+c_2V+c^3UV],\\ c_1 &= (1-z)(b-c)(b-1)[a-1+z(b-c-1)],\\ c_2 &= b(b+1-c)(1-z)(a+bz-cz+2z),\\ c_3 &= c-2b-(a-b)z,\\ C_3(a,b,c) &= abc(c-1)(1-z)^3V,\\ U &= z(a+b-c+1)(a+b-c+2)+ab(1-z),\\ V &= (1-z)(1-a-b+ab)+z(a+b-c-1)(a+b-c-2), \end{aligned} \tag{2.18}$$

with solutions given by

$$\begin{aligned} f_n &= {}_2F_1\left({a+n,\ b+n \atop c-n};z\right),\\ g_n &= \frac{(-z)^n\Gamma(a+1-c+2n)\,\Gamma(b+1-c+2n)}{\Gamma(a+n)\Gamma(b+n)\Gamma(2-c+n)\Gamma(1-c+n)}\times\\ &\qquad {}_2F_1\left({a-c+1+2n,\ b-c+1+2n \atop 2-c+n};z\right). \end{aligned} \tag{2.19}$$

The second solution is obtained from the second term in Eq. (2.7) and applying Eq. (2.10).

Later we will show that f_n is minimal as $n\to+\infty$ on compact domains inside the curve $r=-9+6\sqrt{3}\cos\frac{1}{2}\theta$ with $-\frac{1}{3}\pi\le\theta\le\frac{1}{3}\pi$ (Figure 1) and that it is dominant in compact domains outside the curve. The opposite situation takes place for g_n.

As $n \to -\infty$, which corresponds to case $k = 25$ (i.e., $(-\ -\ +)$), the roles of f_n and g_n are reversed, that is: f_n is dominant (minimal) inside (outside) the curve represented in Figure 1.

Similarly as happened for the case $k = 2$, we can also obtain the condition of the recursion for a recurrence outside our initial target of 26 cases. From Eq. (2.19) we see that the present case is related to the recursion for $(\varepsilon_1\,\varepsilon_2\,\varepsilon_2) = (2\,2\,1)$. In addition, using Eqs. (2.1) and (2.2) we additionally see that case $(2\,2\,1)$ is related to $(2\ -1\,1)$ and $(-1\,2\,1)$.

2.2.3 Basic form $k = 5$

The $(+\,0\,0)$ recursion relation reads

$$A_5(a+n)y_{n-1} + B_5(a+n)y_n + C_5(a+n)y_{n+1} = 0, \tag{2.20}$$

where

$$\begin{aligned} A_5(a) &= (c-a),\\ B_5(a) &= 2a - c - (a-b)z \\ C_5(a) &= a(z-1), \end{aligned} \tag{2.21}$$

with solutions given by

$$\begin{aligned} f_n &= {}_2F_1\left(\begin{matrix} a+n,\ b \\ c \end{matrix};\, z\right), \\ g_n &= (1-z)^{-n}\frac{\Gamma(a+n+1-c)}{\Gamma(a+n)}\,{}_2F_1\left(\begin{matrix} 1-a-n,\ 1-b \\ 2-c \end{matrix};\, z\right), \end{aligned} \tag{2.22}$$

where we have used the first term in the right-hand side of (2.9).

As we will later show, in compact domains inside the circle $|z-1| = 1$, f_n is dominant and g_n is minimal, while the contrary takes place in compact domains outside the circle.

2.2.4 Basic form $k = 6$

The $(+\,0\,-)$ recursion relation reads

$$A_6(a+n, c-n)y_{n-1} + B_6(a+n, c-n)y_n + C_6(a+n, c-n)y_{n+1} = 0, \tag{2.23}$$

where

$$\begin{aligned} A_6(a,c) &= z(a-c)(a-c-1)(b-c)[a + z(b+1-c)],\\ B_6(a,c) &= c[a(a-1)(c-1) + a(a-1)(a+3b-4c+2)z + \\ &\qquad (b-c)(b+1-c)(4a-c-1)z^2 - (a-b)(b-c)(b+1-c)z^3],\\ C_6(a,c) &= -ac(c-1)[a-1+z(b-c)](1-z)^2, \end{aligned} \tag{2.24}$$

with solutions given by

$$f_n = {}_2F_1\left(\begin{matrix} a+n,\ b \\ c-n \end{matrix}; z\right),$$

$$g_n = \frac{\Gamma(a+1-c+2n)\Gamma(b+1-c+n)}{\Gamma(a+b-c+1+2n)\Gamma(1-c+n)}\, {}_2F_1\left(\begin{matrix} a+n,\ b \\ a+b-c+1+2n \end{matrix}; 1-z\right). \tag{2.25}$$

where g_n is selected from the first term of Eq. (2.7).

In this case f_n is never minimal. A satisfactory companion of g_n is always $h_n = g_n - f_n$. The g_n is minimal in compacts domains inside the region between the inner and outer curves in Figure 2. Contrary, h_n is minimal in compact domains inside the complementary region. As $n \to -\infty$ (case $k = 22$, that is $(-\,0\,+)$ or equivalently $(0\,-\,+)$, that is, $k = 16$), the role which was played by h_n for the $(+\,0\,-)$ case is now played by g_n and the role of g_n is played by f_n.

From Eq. (2.25), we observe that information can also be obtained for the recurrence $(1\,0\,2)$ and then (by (2.1), (2.2), and (2.4)) also for the recurrences $(2\,1\,2)$, $(1\,2\,2)$, and $(1\,1\,2)$.

2.2.5 Basic form $k = 13$

The $(0\,0\,+)$ recursion relation reads

$$A_{13}(c+n)y_{n-1} + B_{13}(c+n)y_n + C_{13}(c+n)y_{n+1} = 0, \tag{2.26}$$

where

$$\begin{aligned} A_{13}(c) &= c(c-1)(z-1), \\ B_{13}(c) &= c[c-1-(2c-a-b-1)z], \\ C_{13}(c) &= (c-a)(c-b)z, \end{aligned} \tag{2.27}$$

with solutions given by

$$\begin{aligned} f_n &= {}_2F_1\left(\begin{matrix} a,\ b \\ c+n \end{matrix}; z\right), \\ g_n &= \frac{(-1)^n(1-z)^n\Gamma(c+n)}{\Gamma(c-a-b+1+n)}\, {}_2F_1\left(\begin{matrix} c-a+n,\ c-b+n \\ c-a-b+1+n \end{matrix}; 1-z\right), \end{aligned} \tag{2.28}$$

where we have used the second term in the right-hand side of (2.6).

As we will show next, f_n is minimal in compact domains inside the region $\Re z < 1/2$, where g_n is dominant. Contrary, in compact domains inside the region $\Re z > 1/2$ f_n is dominant and g_n is minimal.

When $n \to -\infty$ (that is, for the case $k = 15$, corresponding to $(0\,0\,-)$) f_n is a dominant solution both for $\Re z < 1/2$ and $\Re z > 1/2$. The satisfactory

companion solution for f_n (the minimal solution) is chosen differently for the cases $\Re z < 1/2$ and $\Re z > 1/2$ (see §5.1).

3 Domains for minimal and dominant solutions

Perron's theorem (see [10, Appendix B]) gives in the case of finite limits the following results. Let α and β denote

$$\alpha := \lim_{n\to\infty} \frac{B_n}{C_n}, \quad \beta := \lim_{n\to\infty} \frac{A_n}{C_n}. \tag{3.1}$$

Let t_1 and t_2 denote the zeros of the characteristic polynomial $t^2+\alpha t+\beta = 0$. If $|t_1| \neq |t_2|$, then the difference equation (1.3) has two linear independent solutions f_n and g_n with the properties

$$\frac{f_{n+1}}{f_n} \sim t_1, \quad \frac{g_{n+1}}{g_n} \sim t_2. \tag{3.2}$$

If $|t_1| = |t_2|$, then

$$\limsup_{n\to\infty} |y_n|^{\frac{1}{n}} = |t_1| \tag{3.3}$$

for any non-trivial solution y_n of (1.3).

In the following subsections, we give for the five basic forms the domains in the $z-$plane where $|t_1| \neq |t_2|$. In these domains there is a true distinction between the two solutions of (1.3). If $|t_1| > |t_2|$ then the solution f_n that satisfies the relation in (3.2) is a maximal solution and g_n is the minimal solution. On the curves where $|t_1| = |t_2|$ the two solutions are neither dominant nor minimal, and recursion in forward or backward direction is not unstable.

For all five basic forms the ratios A_n/C_n and B_n/C_n of the difference equation (1.3) tend to finite limits as $|n| \to \infty$. Interestingly, in all cases these limits are functions of z, and they are not depending on the parameters a, b or c.

3.1 The domains for basic form $k = 2$

The limits α and β of (2.11) are

$$\alpha = -\frac{2(z+1)}{(1-z)^2}, \quad \beta = \frac{1}{(1-z)^2}. \tag{3.4}$$

The zeros of the characteristic polynomial are

$$t_1 = \frac{1}{(1-\sqrt{z})^2}, \quad t_2 = \frac{1}{(1+\sqrt{z})^2}. \tag{3.5}$$

The equation $|t_1| = |t_2|$ holds when $z \le 0$, otherwise $|t_1| > |t_2|$.

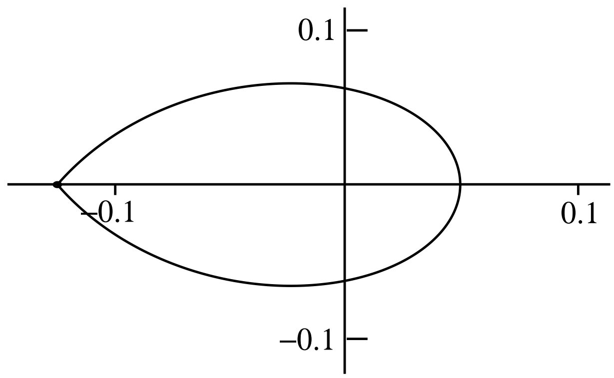

Figure 1: The curve $|t_1| = |t_2|$ for the basic form $k = 3$.

3.2 The domains for basic form $k = 3$

The limits α and β of (2.17) are

$$\alpha = \frac{8z^2 + 20z - 1}{(1-z)^3}, \quad \beta = -\frac{16z}{(1-z)^3}. \tag{3.6}$$

The zeros of the characteristic polynomial are

$$t_1 = \frac{1 - 20z - 8z^2 + (8z+1)^{\frac{3}{2}}}{2(1-z)^3}, \quad t_2 = \frac{1 - 20z - 8z^2 - (8z+1)^{\frac{3}{2}}}{2(1-z)^3}. \tag{3.7}$$

We write this in the form

$$\begin{aligned} t_1 &= \frac{27 - 18w^2 - w^4 + 8w^3}{16(1-z)^3} = \frac{32(1+w)}{(3+w)^3}, \\ t_2 &= \frac{27 - 18w^2 - w^4 - 8w^3}{16(1-z)^3} = \frac{32(1-w)}{(3-w)^3}, \end{aligned} \tag{3.8}$$

where $w = \sqrt{8z+1}$. To find the curve in the $w-$plane defined by $|t_1| = |t_2|$, we write $w = re^{i\theta}$. This gives the curve described by

$$r = -9 + 6\sqrt{3}\cos\tfrac{1}{2}\theta, \quad -\tfrac{1}{3}\pi \le \theta \le \tfrac{1}{3}\pi. \tag{3.9}$$

In Figure 1 we show this curve in the $z-$plane. In the domain interior to this curve we have $|t_1| > |t_2|$.

3.3 The domains for basic form $k = 5$

The limits α and β of (2.20) are

$$\alpha = \frac{z-2}{1-z}, \quad \beta = \frac{1}{1-z}. \tag{3.10}$$

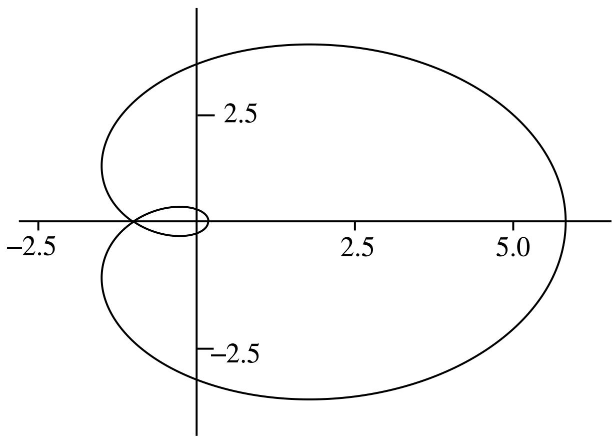

Figure 2: The curve $|t_1| = |t_2|$ for the basic form $k = 6$.

The zeros of the characteristic polynomial are

$$t_1 = 1, \quad t_2 = \frac{1}{1-z}. \tag{3.11}$$

The equation $|t_1| = |t_2|$ holds when $|1-z| = 1$, which defines a circle with centre $z = 1$ and radius 1. Inside the circle we have $|t_2| > |t_1|$.

3.4 The domains for basic form $k = 6$

The limits α and β of (2.23) are

$$\alpha = -\frac{z^2 - 6z + 1}{(1-z)^2}, \quad \beta = -\frac{4z}{(1-z)^2}. \tag{3.12}$$

The zeros of the characteristic polynomial are

$$t_1 = 1, \quad t_2 = -\frac{4z}{(1-z)^2}. \tag{3.13}$$

To find the curve defined by $|t_1| = |t_2|$, we write $z = re^{i\theta}$. This gives the curve described by

$$r = 2 + \cos\theta \pm \sqrt{\cos 2\theta + 4\cos\theta + 3}, \quad -\pi \le \theta \le \pi. \tag{3.14}$$

Both signs give a closed loop with common point -1. In Figure 2 we show this curve in the $z-$plane. In the domain interior to the inner curve we have $|t_1| > |t_2|$; between the inner curve and the outer curve we have $|t_1| < |t_2|$, and outside the outer curve $|t_1| > |t_2|$.

3.5 The domains for basic form $k = 13$

The limits α and β of (2.26) are

$$\alpha = -\frac{2z-1}{z}, \quad \beta = \frac{z-1}{z}. \tag{3.15}$$

The zeros of the characteristic polynomial are

$$t_1 = 1, \quad t_2 = \frac{z-1}{z}. \tag{3.16}$$

The equation $|t_1| = |t_2|$ holds when $\Re z = \frac{1}{2}$. When $\Re z > \frac{1}{2}$ we have $|t_1| > |t_2|$.

4 Asymptotics for minimal and dominant solutions

In some cases we simply use the power series in (1.2), which provides an asymptotic expansion for large c. In some other cases we can use connection formulas for transforming the Gauss function to the case for large c. We also use the integral representations

$${}_2F_1\left({a,\ b \atop c};\, z\right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\, dt, \tag{4.1}$$

$${}_2F_1\left({a,\ b \atop c};\, z\right) = \frac{\Gamma(c)\,\Gamma(1+b-c)}{2\pi i\,\Gamma(b)} \int_0^{(1^+)} t^{b-1}(t-1)^{c-b-1}(1-tz)^{-a}\, dt, \tag{4.2}$$

where in the first integral $\Re c > \Re b > 0$ and in the second one $\Re b > 0$. In the second integral the contour starts and terminates at $t = 0$ and encircles the point $t = 1$ in the positive direction. The point $t = 1/z$ should be outside the contour. The many-valued functions of the integrand assume their principal branches: the phase of $(1 - tz)$ tends to zero when $z \to 0$, and the phases of t and $(t-1)$ are zero at the point where the contour cuts the real positive axis (at the right of $t = 1$). To prove (4.2) integrate (if $\Re c > \Re b > 0$) along the interval $(0, 1)$ with proper choices of the branches of the many-valued functions at the upper and lower sides of $(0, 1)$, and obtain (4.1); see also [9, p. 111]. The integral in (4.1) is Euler's well-known standard representation.

We use saddle-point methods (see [11]) to obtain asymptotic estimates of these integrals when one of more parameters are large. We omit details of the saddle-point analysis because we only need to obtain the main terms in the asymptotic estimates for identifying minimal and dominant solutions. We assume that z is fixed and properly inside the domains described in Section 3.

For recent papers on uniform asymptotic expansions of hypergeometric functions, see [5], [6] and [7].

4.1 Asymptotics for basic form $k = 2$

For f_n of (2.13) we use (4.2) and we obtain

$$f_n = \frac{\Gamma(c)\,\Gamma(1+b+n-c)}{2\pi i\,\Gamma(b+n)} \int_0^{(1^+)} t^{b-1}(t-1)^{c-b-1}(1-tz)^{-a}\, e^{n\phi(t)}\, dt, \quad (4.3)$$

where $\phi(t) = \ln t - \ln(t-1) - \ln(1-tz)$. The saddle-points are found by putting $\phi'(t) = 0$, giving $t_\pm = \pm 1/\sqrt{z}$. If $z \in (0,1)$ we have $t_+ > 1$ and the saddle-point contour starts at $t = 0$, turns around $t = 1$ through t_+, and returns to $t = 0$. The dominant term in the asymptotic estimate is $e^{\phi(t_+)} = t_1^n$ (see (3.5)). This also holds for complex values of z.

For g_n of (2.13) we use (4.1) and we obtain

$$\begin{aligned} g_n = {} & \frac{\Gamma(a+n+1-c)\,\Gamma(b+n+1-c)}{\Gamma(a+n+1-c)\Gamma(b+n)} \times \\ & \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-t(1-z))^{-a}\, e^{n\psi(t)}\, dt, \end{aligned} \quad (4.4)$$

where $\psi(t) = \ln t + \ln(1-t) - \ln(1-t(1-z))$. The saddle-points are $t_+ = \sqrt{t_2}$, $t_- = \sqrt{t_1}$. When $z \in (0,1)$ we have $t_+ \in (0,1)$ and this point gives the dominant contribution. The dominant term in the asymptotic estimate is $e^{\psi(t_+)} = t_2^n$. This also holds for complex values of z.

When z is not a negative real number we infer that for g_n the main term in the asymptotics is t_2^n and that for f_n the main term is t_1^n. We conclude that in compact domains that do not contain points of $(-\infty, 0]$, g_n of (2.13) is the minimal solution and f_n is a dominant solution of (2.11).

4.2 Asymptotics for basic form $k = 3$

For f_n of (2.19) we use (4.2) and we obtain

$$f_n = \frac{\Gamma(c-n)\,\Gamma(1+b-c+2n)}{2\pi i\,\Gamma(b+n)} \int_0^{(1^+)} t^{b-1}(t-1)^{c-b-1}(1-tz)^{-a}\, e^{n\phi(t)}\, dt, \quad (4.5)$$

where $\phi(t) = \ln t - 2\ln(t-1) - \ln(1-tz)$. The saddle-points are $t_\pm = (1 \pm w)/(4z)$, $w = \sqrt{1+8z}$. If $z \in (0,1)$ we have $t_+ > 1$ and the saddle-point contour starts at $t = 0$, turns around $t = 1$ through t_+, and returns to $t = 0$. The dominant term in the asymptotic estimate is $e^{\phi(t_+)} = (-1)^n 2^{-2n} t_2^n$ (see (3.7)). This also holds for complex values of z.

For g_n of (2.19) we again use (4.2) and we obtain

$$\begin{aligned} g_n = {} & \frac{(-z)^n\,\Gamma(a+1-c+2n)}{2\pi i\,\Gamma(a+n)\Gamma(1-c+n)} \times \\ & \int_0^{(1+)} t^{b-c}(t-1)^{-b}(1-tz)^{-a+c-1}\, e^{n\psi(t)}\, dt, \end{aligned} \quad (4.6)$$

where $\psi(t) = 2\ln t - \ln(t-1) - 2\ln(1-tz)$. The saddle-points are $t_\pm = (-1 \pm w)/(2z)$. When $z \in (0,1)$ we have $t_+ \in (1,2)$ and the saddle-point contour is similar as for f_n. The dominant term in the asymptotic estimate is $e^{\psi(t_+)} = (-z)^{-n}2^{-2n}t_1^n$ (see (3.7)). This also holds for complex values of z.

Taking into account the asymptotics of the gamma function in front of the integrals in (4.5) and (4.6), we infer that the dominant terms in the asymptotic estimate of f_n is t_2^n; and for g_n it is t_1^n. We conclude that in compact domains interior to the curve of Figure 1, g_n of (2.19) is a dominant solution and f_n is the minimal solution of (2.17). In compact domains exterior to this curve the roles of f_n and g_n are interchanged.

4.3 Asymptotics for basic form $k = 5$

For f_n of (2.22) we use (4.1) and we obtain

$$f_n = \frac{\Gamma(c)}{\Gamma(b)\,\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a-n}\,dt, \tag{4.7}$$

If $|1-z| < 1$ the main contributions to the integral come from $t-$values near the end-point 1. In that case the dominant term in the asymptotic estimate is t_2^n (see (3.11)). If $|1-z| > 1$ the main contributions to the integral come from $t-$values near the end-point 0. In that case $f_n = \mathcal{O}(1/n)$ as $n \to \infty$.

For g_n of (2.22) we use (2.3), which gives

$$g_n = (1-z)^{1+b-c-n}\frac{\Gamma(a+n+1-c)}{\Gamma(a+n)}\,{}_2F_1\left({1+a-c+n,\ 1-b \atop 2-c};\frac{z}{1-z}\right). \tag{4.8}$$

We can use the same method as for f_n, and conclude that in compact domains inside the disk $|z-1| < 1$, g_n of (2.22) is a minimal solution and f_n is a dominant solution of (2.20), respectively corresponding with t_1 and t_2 of (3.11). In compact domains outside the disk the roles of f_n and g_n are interchanged.

4.4 Asymptotics for basic form $k = 6$

For g_n of (2.25) we use (2.1) and (4.1), and obtain

$$g_n = \frac{\Gamma(a+1-c+2n)}{\Gamma(a+n)\Gamma(1-c+n)} \int_0^1 t^{a-1}(1-t)^{b-c}[1-(1-z)t]^{-b}e^{n\phi(t)}\,dt \tag{4.9}$$

where $\phi(t) = \ln t + \ln(1-t)$. The point $t = \frac{1}{2}$ gives the main contribution, and the dominant term of the integral is 4^{-n}.

Next we consider $h_n := g_n - f_n$. From (2.28), (2.7) and (4.1) it follows that

$$\begin{aligned} h_n = {} & \frac{z^{1-c+n}\,\Gamma(c-n-1)\Gamma(a+1-c+2n)\Gamma(2-c+n)}{\Gamma(a+n)\Gamma(b)\Gamma(1-c+n)\Gamma(1-b)} \times \\ & \int_0^1 t^{b-c}(1-t)^{-b}[1-t(1-z)]^{-a+c-1}\,e^{n\psi(t)}\,dt, \end{aligned} \tag{4.10}$$

where $\psi(t) = \ln t - 2\ln(1-tz)$. The end-point $t=1$ gives the dominant contribution $z^n/(1-z)^{2n}$ to the integral.

Taking into account the contribution from the gamma functions in (4.9) and (4.10), it follows that g_n corresponds with t_1 of (3.13), for all z, and h_n with t_2.

Because $f_n = g_n - h_n$, this function is never a minimal solution of (2.23). In compact domains interior to the inner curve and outside the outer curve (see Figure 2), g_n is a dominant solution and h_n the minimal solution; in compact domains between the inner and the outer curve the roles of g_n and h_n are interchanged.

4.5 Asymptotics for basic form $k = 13$

For f_n of (2.28) we have the estimate $f_n = 1+\mathcal{O}(1/n)$ for all z. For g_n we apply (2.4), and obtain

$$g_n = \frac{(1-z)^n}{z^n}\frac{z^{1-c}\,\Gamma(c+n)}{\Gamma(c-a-b+1+n)}\,{}_2F_1\left(\begin{matrix}1-b,\ 1-a\\ c-a-b+1+n\end{matrix};1-z\right). \quad (4.11)$$

We conclude that in compact domains of $\Re z < \frac{1}{2}$, f_n is the minimal solution and g_n is a dominant solution. In compact domains of $\Re z > \frac{1}{2}$ the roles of f_n and g_n are interchanged.

5 The cases $k = 15, k = 16, k = 25$

The cases $k = 15, k = 16, k = 25$ in Table 1 are special because we refer to these as ‘change signs in other cases’, and we do not consider them as basic forms. The recursion relations for the cases $k = 15, k = 16, k = 25$ are the same as those for $k = 13, k = 12, k = 3$, respectively, when we recur backwards, that is to $-\infty$. The zeros t_1 and t_2 of the characteristic polynomial do not change when we change the recursion direction. In this section we give linearly independent satisfactory solutions for the cases $k = 15, k = 16, k = 25$.

5.1 The case $k = 15$

We verify if the solutions given for the recursion relation of the case $k = 13$ can be used as satisfactory linear independent solutions for the same recursion relation used in backward direction.

The asymptotics for of f_n of (2.28) for large positive n follows easily from the power series. For negative n we can use (2.7) and for the term with Q we use also (2.4). This gives

$$\begin{aligned} f_{-n} = {}_2F_1\left(\begin{matrix}a,\ b\\ c-n\end{matrix};z\right) &= P\,{}_2F_1\left(\begin{matrix}a,\ b\\ a+b-c+n+1\end{matrix};1-z\right) \\ &\quad - Q\,z^{1-c+n}(1-z)^{c-n-a-b}\,{}_2F_1\left(\begin{matrix}1-a,\ 1-b\\ 2-c+n\end{matrix};z\right), \end{aligned} \quad (5.1)$$

where P and Q are as in (2.8) with n replaced with $-n$. Again the power series can be used for obtaining the asymptotics, and it follows that for f_{-n} of (5.1) we have $f_{-n} = 1 + \mathcal{O}(1/n)$ as $n \to +\infty$, when $|z/(1-z)| < 1$; otherwise, the dominant term in the asymptotic behaviour is $|z/(1-z)|^n$.

For the second solution g_n of (2.28) we obtain, after using (2.4),

$$g_{-n} = \frac{(-1)^n z^{1-c}\Gamma(c-n)}{\Gamma(c-a-b+1-n)} \left(\frac{z}{1-z}\right)^n {}_2F_1\left({1-b,\ 1-a \atop c-a-b+1-n}; 1-z\right). \quad (5.2)$$

We can verify as for the first $F-$function in (5.1) that the $F-$function in (5.2) has dominant term $[(1-z)/z]^n$ when $|z/(1-z)| < 1$; otherwise, the dominant term in the asymptotic behaviour is algebraic in n. Because of the extra factor $[z/(1-z)]^n$ in front of the $F-$function in (5.2), it follows that f_{-n} and g_{-n} are minimal or dominant in the same $z-$domain. Hence the $\{f_n, g_n\}$ of (2.28) does not constitute a satisfactory pair of linearly independent solutions for the recursion (2.27) in backward direction. As a consequence, we cannot use case $k = 13$ in backward direction for the present case $k = 15$.

The second solution g_n of (2.28) has been selected by considering the second term in (2.6). When instead we consider the first term, using (2.10), with c replaced with $c - n$, we can take as another solution of (2.27) the function h_n, where

$$h_{-n} = \frac{\Gamma(n+1-c+a)\Gamma(n+1-c+b)}{\Gamma(n+1-c)\Gamma(n+1-c+a+b)} {}_2F_1\left({a,\ b \atop n+1-c+a+b}; 1-z\right). \quad (5.3)$$

The $F-$function is $1 + \mathcal{O}(1/n)$ for large positive n in compact $z-$domains. Hence, h_n can be used as a proper second solution together with f_n for the recursion relation of the case $k = 13$ in backward direction when $|z/(1-z)| > 1$. We still need another function for the complementary domain. We consider for this the connection formula (2.9) use the first term in the right-hand side, and (2.10), and we take as a new solution of (2.27) the function j_n, where

$$\begin{aligned} j_{-n} = (-1)^n \frac{\Gamma(a-c+n+1)\Gamma(b-c+n+1)}{\Gamma(2-c+n)\Gamma(1-c+n)} \times \\ \left(\frac{z}{1-z}\right)^n {}_2F_1\left({1-a,\ 1-b \atop 2-c+n}; z\right). \end{aligned} \quad (5.4)$$

The $F-$function is $1 + \mathcal{O}(1/n)$ for large positive n in compact $z-$domains. Hence, j_n can be used as a proper second solution together with f_n for the recursion relation of the case $k = 13$ in backward direction when $|z/(1-z)| < 1$.

Summarizing, for the present case $k = 15$ we can use the recursion (2.27) of basic form $k = 13$ in backward direction with f_n of (2.28) and h_n of (5.3) for $z-$values in compact domains of the half-plane $\Re z > \frac{1}{2}$ where f_n is a dominant and h_n is the minimal solution. In compact domains of $\Re z < \frac{1}{2}$ we can use f_n together with j_n of (5.4), with f_n as a dominant and j_n as the minimal solution.

5.2 The case $k = 16$

We verify if the solutions given for the recursion relation of the case $k = 6$ can be used as linear independent solutions for the same recursion relation used in backward direction.

The function f_n of the case $k = 6$ given in (2.25) is $1 + \mathcal{O}(1/n)$ as $n \to -\infty$. This easily follows when we use the integral representation in (4.1). Using this integral g_n of (2.25) it follows that for g_n the dominant term is t_2^{-n}, where t_2 is given in (3.13).

It follows that for the present case $k = 16$ we can use the recursion of basic form $k = 6$, that is, (2.23), in backward direction with f_n and g_n of (2.25). Inside compact domains interior to the inner curve (around the origin) of Figure 2 and exterior to the outer curve f_n is a dominant and g_n is the minimal solution. In compact domains between the two curves the roles of f_n and g_n are interchanged.

5.3 The case $k = 25$

We verify if the solutions given for the recursion relation of the case $k = 3$ can be used as linear independent solutions for the same recursion relation used in backward direction.

When we apply (2.4) to both f_n and g_n of (2.19), we see that the asymptotics for $n \to -\infty$ follows from that of $n \to +\infty$, when the roles of f_n and g_n are interchanged.

We conclude that in compact domains interior to the curve of Figure 1, f_n is a dominant solution and g_n is the minimal solution of of (2.17) when used in backward direction. In compact domains exterior to this curve the roles of f_n and g_n are interchanged.

6 Numerical examples

The power series (1.2) is very useful for numerical computations for z properly inside the unit disk. Transformations and connection formulas as in (2.2), (2.3), (2.6) and (2.9) can be used to cover large parts of the complex $z-$plane. Other connection formulas are available with other $z-$arguments and in fact for the computation of the Gauss function we can use power series with powers of

$$z, \quad 1-z, \quad \frac{1}{z}, \quad \frac{z-1}{z}, \quad \frac{1}{1-z}, \quad \frac{z}{z-1}. \tag{6.1}$$

For numerical computations we need convergence conditions like

$$|z| < \rho, \ |1-z| < \rho, \ \left|\frac{1}{z}\right| < \rho, \ \left|\frac{z-1}{z}\right| < \rho, \left|\frac{1}{1-z}\right| < \rho, \ \left|\frac{z}{z-1}\right| < \rho, \tag{6.2}$$

with $0 < \rho < 1$.

Not all points in the $z-$plane satisfy one of these inequalities for a given number ρ. In Figure 3 we take $\rho = \frac{3}{4}$. In the dark area at least one of the above

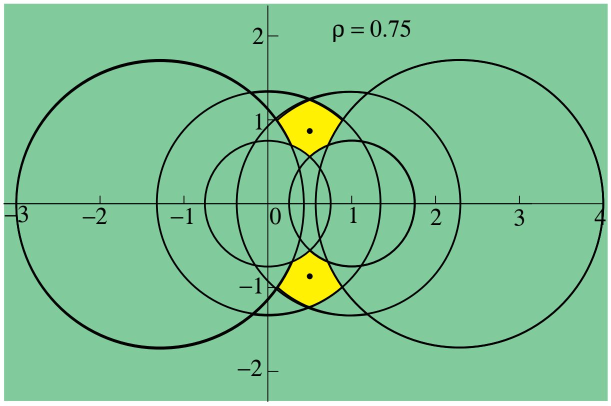

Figure 3: In the light domains around the points $e^{\pm\pi i/3}$ none of the inequalities of (6.2) is satisfied.

inequalities is satisfied. In the light areas, "around" the points $e^{\pm\pi i/3}$, none of these inequalities is satisfied. By choosing ρ closer to unity these light domains become smaller.

For certain combinations of the parameters a, b and c, the connection formulas become numerically unstable. For example, if $c = a + b$, the relation in (2.6) is well-defined, although two gamma functions are infinite. By using a limiting procedure the value of ${}_2F_1(a, b; a + b; z)$ can be found. For c close to $a + b$ numerical instabilities occur when using (2.6). See [2] for many examples and details.

Other instabilities in the evaluation of the power series (1.2) may arise for large values of a and b.

In [10, p. 71] an example is given how to compute a Gauss function with argument $z = e^{\pi i/3}$, the point that is excluded from the convergence domains shown in Figure 3. Wimp considers the computation of

$$ {}_2F_1\left(\begin{matrix}\frac{2}{3},\ 1\\ \frac{4}{3}\end{matrix};\, e^{\pi i/3}\right) = \frac{2\pi e^{\pi i/6}\Gamma(\frac{1}{3})}{9[\Gamma(\frac{2}{3})]^2}, \tag{6.3} $$

by using a Miller algorithm for the hypergeometric functions

$$ f_n = {}_2F_1\left(\begin{matrix}n+a,\ n+b\\ 2n+c\end{matrix};\, z\right), \quad n = 0, 1, 2, \ldots\ . \tag{6.4} $$

This recursion type was not initially included in the group of 26 discussed in the present paper. However as, discussed in §2.2.1, this recurrence can be related to the case $k = 2$ to conclude that f_n is minimal. Therefore, Miller's algorithm can be applied when a sum rule is provided, as done in [10, p. 71].

We can also use basic form $k = 13$, with pure $c-$recursion. From §3.5 and §4.5 it follows that for the point $z = e^{\pi i/3} = \frac{1}{2} + \frac{1}{2}i\sqrt{3}$ the solutions of the recursion relation (2.26) are neither dominant nor minimal. We use backward recursion for ${}_2F_1(a, b; c + n; z)$ with two starting values for $n = 29$ and $n = 30$. With these large values of $c+n$ the power series converges fast. In 15D arithmetic we have computed the value of (6.3) with a relative error 2×10^{-14}. The exact value is

$$0.883319375142724... + 0.509984679019064...i\ .$$

With recursion we obtain

$$0.883319375142719 + 0.509984679019039i\ .$$

In [10, p. 72] another example of the Miller algorithm is discussed for the basic form $k = 13$ (pure $c-$recursion).

Several algorithms based on recursion relations for special cases of the Gauss functions have been published, in particular for computing Legendre functions. For recent papers, see [3], [4] and [8].

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